

# A Simple Optimal Power Flow Model with Energy Storage

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**Abstract**—The integration of renewable energy, such as wind power, into the electric grid is difficult because of the source intermittency and the large distance between generation sites and users. This difficulty can be overcome through a transmission network with large-scale storage that not only transports power, but also mitigates against fluctuations in generation and supply. We formulate an optimal power flow problem with storage as a finite-horizon optimal control problem. We prove, for the special case with a single generator and a single load, that the optimal generation schedule will cross the time-varying demand profile at most once, from above. This means that the optimal policy will generate more than demand initially in order to charge up the battery, and then generate less than the demand and use the battery to supplement generation in final stages. This is a consequence of the fact that the marginal storage cost-to-go decreases in time.

## I. INTRODUCTION

The optimal power flow (OPF) problem is to optimize a certain objective over power network variables under certain constraints. The variables may include real and reactive power outputs, bus voltages and angles; the objective may be the minimization of generation cost or maximization of user utilities; and the constraints may be bounds on voltages or power levels, or that the line loading not exceeding thermal or stability limits. OPF has been studied for over half a century; see recent works in e.g. [1], [2], [3], [4] and surveys in e.g. [5], [6]. Its history is roughly a continuous application of more and more sophisticated optimization techniques [6]. Most of the models involve static optimization as, without large scale storage, power supply and demand must be matched exactly at all times, and therefore OPF can be solved in isolation from one period to the next. In this paper, we formulate a simple OPF model with storage and study how storage allows optimization of power generation across multiple time periods.

The model is motivated by the intensifying trend to deploy renewable energy such as wind or solar power. In the state of California, peak demand for power in 2003 reached 52 GW, with projections for the year 2030 exceeding 80 GW [7], [8]. With fossil fuel and nuclear plants retiring in the next few decades, and a required 15% reserve margin, an additional 60 GW of new generation capacity will be needed by 2030 [8]. In 2006, Southern California Edison, the primary electricity utility company for the southern California, signed a contract

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to provide 1.5 GW of power to its customers from wind projects in the Tehachapi area in California [9]. Not only is renewable energy more environmentally friendly, but its potential is high. It has been estimated that if 20% of the energy that could be harvested from wind farms across the globe were used, all of the world's electric demands could be met several times over [10]. There are however two major challenges in integrating wind or solar power into the current system. First, their output fluctuates widely, rapidly, and randomly, a great operational hurdle. Second, the geographical locations of wind or solar farms are inevitably far from the load centers. These two problems calls for large-scale storage that can absorb short-term fluctuations and transmission capacity that not only transports power from generation to load, but can also provide spatial diversity in generation to mitigate intermittency of renewable sources. See e.g. [11] for an early simulation study on how battery can help with regulation and peak-shaving in economic dispatch.

In Section II, we present our OPF model with storage and formulate the optimal generation and battery charging schedule as a finite-horizon optimal control problem. In Section III, we consider the special case with a single generator and a single load (SGSL case). Without battery, the generation must be exactly equal to the load. With battery, we prove that the optimal generation schedule will cross the time-varying demand profile at most once, from above. This means that the optimal policy will generate more than demand initially in order to charge up the battery, and then generate less than the demand and use the battery to supplement generation in final stages. This is a consequence of the fact that the marginal storage cost-to-go decreases in time. In Section IV, we discuss the optimality conditions for the general network case and present some numerical examples that suggest the intuition in the SGSL case might generalize to the network case. We conclude in Section V.

## II. MODEL AND PROBLEM FORMULATION

We now present a simple OPF model with energy storage and time-varying generation costs and demands. The model ignores reactive power and makes other simplifying assumptions. Our goal is to understand the impact of storage on optimal generation schedule.

Consider a set  $\mathcal{G}$  of generation nodes/buses connected to a set  $\mathcal{D}$  of demand nodes/buses by a transmission network. Let  $\mathcal{N} = \mathcal{G} \cup \mathcal{D}$  be the set of all nodes. The transmission network is modeled by the admittance matrix  $Y$ , where  $Y_{ij} = Y_{ji}$  is the admittance between nodes  $i$  and  $j$ . If nodes  $i$  and  $j$  are not directly connected,  $Y_{ij} = 0$ .

The (real) power flow from node  $i$  to node  $j$  at time  $t$  is  $V_i V_j Y_{ij} \sin(\theta_i(t) - \theta_j(t))$  for  $i \neq j \in \mathcal{N}$ . In this paper, we will assume  $|\theta_i(t) - \theta_j(t)|$  is small and approximate  $\sin(\theta_i(t) - \theta_j(t))$  by  $\theta_i(t) - \theta_j(t)$ . The amount of power delivered over link  $(i, j)$  is limited by thermal effects and network stability [12, chapter 4]. Current flows produce power losses alongside heat generation, reducing transmission efficiency and creating line sag from temperature rises. We capture these constraints as

$$V_i V_j Y_{ij} (\theta_i(t) - \theta_j(t)) \leq \bar{q}_{ij}(t), \quad i \neq j \in \mathcal{N} \quad (1)$$

where  $\bar{q}_{ij}(t)$  represents the line capacity from nodes  $i$  to  $j$ . From Kirchoff's laws, the net power export from node  $i$  at time  $t$  is given by

$$q_i(t) = \sum_{j \in \mathcal{N}} V_i V_j Y_{ij} (\theta_i(t) - \theta_j(t)), \quad i \in \mathcal{N} \quad (2)$$

If  $q_i(t)$  is positive, node  $i$  supplies power at time  $t$ ; otherwise, it consumes power at time  $t$ .

Each node  $i \in \mathcal{D}$  demands a fixed amount  $d_i(t)$  of power at time  $t$  that must be met by supplies from the generation nodes

$$q_i(t) = -d_i(t), \quad i \in \mathcal{D} \quad (3)$$

Each generation node  $i \in \mathcal{G}$  has both a generator that produces  $g_i(t)$  amount of power at time  $t$  and a battery that can charge or discharge  $r_i(t)$  amount of power at time  $t$ . The net power export  $q_i(t)$  from generator node  $i$  at time  $t$  consists of the power from the node's generator and battery:

$$q_i(t) = g_i(t) + r_i(t), \quad i \in \mathcal{G} \quad (4)$$

$$g_i(t) \geq 0, \quad i \in \mathcal{G} \quad (5)$$

Note that  $r_i(t)$  can either be negative (battery is charging) or positive (battery is discharging). The battery energy level  $b_i(t)$  at node  $i \in \mathcal{G}$  evolves according to

$$b_i(t) = b_i(t-1) - r_i(t), \quad i \in \mathcal{G} \quad (6)$$

with given initial energy level  $b_i(0) \geq 0$ . Battery storage is bounded by a minimum and a maximum capacity

$$0 \leq b_i(t) \leq B_i \quad i \in \mathcal{G} \quad (7)$$

We assume the cost  $c_i(g_i, t)$  of generation at generator  $i \in \mathcal{G}$  is a function of the amount of generated power  $g_i$  and time  $t$ . Additionally, there is a battery cost  $h_i(b_i, r_i)$  as a time-invariant function of energy level  $b_i$  and the power draw  $r_i$ .

The classical OPF problem without storage determines the voltage angles  $\theta_i(t)$ ,  $i \in \mathcal{N}$ , and power generations  $q_i(t)$ ,  $i \in \mathcal{G}$ , so as to minimize the total generation cost. Without storage, there is no correlation across time, and therefore the optimization problem is static and can be solved in isolation from one period to the next. The transmission network  $Y$  allows optimization across space. Storage allows optimization across time as well, i.e., charge when (and where) the cost of generation is low and discharge when

it is high. The optimization problems in each period thus become coupled, yielding an optimal control problem.

Given admittance matrix  $Y$ , initial battery levels  $b_i(0) \geq 0$  and storage capacities  $B_i \geq 0$  for  $i \in \mathcal{G}$ , link capacities  $\bar{q}_{ij}$  for  $i, j \in \mathcal{N}$ , and demand profiles  $d_i(t)$  for  $i \in \mathcal{D}$ ,  $t = 1, \dots, T$ , the OPF problem with energy storage is

**OPF-S**

$$\begin{aligned} \min_{\theta, q, r, g, b} \quad & \sum_{t=1}^T \sum_{i \in \mathcal{G}} \{c_i(g_i(t), t) + h_i(b_i(t), r_i(t))\} \quad (8) \\ \text{s.t.} \quad & (1), (2), (3), (4), (5), (6), (7) \quad (9) \end{aligned}$$

where  $t = 1, \dots, T$  in constraints (1)-(7). Note that in this model only the generation costs  $c_i(g_i, t)$  and demand profiles  $d_i(t)$  are time-varying.

In this paper, we restrict ourselves to the quadratic costs

$$c_i(g_i(t), t) := \frac{1}{2} \gamma_i(t) g_i^2(t) \quad (10)$$

where  $\gamma_i(t)$  models the time-varying nature of the generation costs. The convexity of  $c_i$  in  $g_i$  reflects the decreased efficiency of the generator when producing very high amounts of power [12, chapter 11]. We also assume a battery cost that is independent of the power draw  $r_i$ , but dependent only on the energy level  $b_i$ :

$$h_i(b_i, r_i) = h_i(b_i) \quad (11)$$

where  $\alpha_i > 0$ . We will require in assumption A0 that  $h_i(b_i)$  is strictly decreasing in  $b_i$ , penalizing low energy level, e.g.,  $h_i(b_i) = \alpha_i(B_i - b_i)$  for some  $\alpha_i > 0$ .

**Notations.** We will use  $h'(b(t))$  and  $\frac{dh}{db}(b(t))$  interchangeably. For both primal variables, such as  $g_i(t), b_i(t)$ , and their dual variables, such as  $\bar{b}_i(t), \underline{b}_i(t), \bar{b}_i(t)$ , a  $*$  denotes their value at optimality, e.g.,  $g_i^*(t), b_i^*(t), \bar{b}_i^*(t), \underline{b}_i^*(t), \bar{b}_i^*(t)$ . For  $x \in \mathbb{R}$ ,  $[x]^+ := \max(x, 0)$ .

### III. SINGLE GENERATOR SINGLE LOAD

In this section we solve the simplest case of a single generator (with a battery) connected to a single load. Removal of the network structure allows us to completely characterize the solution and make transparent the effect of storage on the optimal generation schedule and the optimal charge-discharge schedule of the battery. We expect the basic insight to generalize to the network case, the discussion of which is in the next section.

#### A. SGSL model

From (2), (3), and (4) we have

$$g(t) + r(t) = d(t) \quad (12)$$

This simplifies the problem OPF-S into the following

$$\begin{aligned}
\min \quad & \sum_{t=1}^T c(g(t), t) + h(b(t)) \\
\text{s. t.} \quad & b(t) = b(t-1) - d(t) + g(t) \quad [\text{dual var: } \tilde{b}(t)] \\
& g(t) \geq 0 \quad [\text{dual var: } \hat{\lambda}(t)] \\
& b(t) \geq 0 \quad [\text{dual var: } \underline{b}(t)] \\
& B - b(t) \geq 0 \quad [\text{dual var: } \bar{b}(t)] \\
\text{for} \quad & t = 1, \dots, T
\end{aligned}$$

where  $b(0) \geq 0$  is given, and the cost functions  $c$  and  $h$  are given by (10) and (11) respectively. The capacity constraint (1) degenerates into  $d(t) \leq \bar{q}$  where  $\bar{q}$  is the capacity of the link connecting the generator to the load. For feasibility, we assume that this constraint is satisfied.

The problem OPF-S is a convex program and therefore the KKT condition is both necessary and sufficient for optimality. Differentiating with respect to  $b(t)$ , the KKT condition implies

$$\frac{dh}{db}(b^*(t)) + \tilde{b}^*(t) - \tilde{b}^*(t+1)\mathbf{1}(t < T) = \underline{b}^*(t) - \bar{b}^*(t)$$

where  $\mathbf{1}$  denotes the indicator function. This defines a recursion on  $\tilde{b}^*(t)$  whose solution, for  $t = 1, \dots, T$ , is

$$\tilde{b}^*(t) = H^*(t) + B^*(t) \quad (13)$$

with  $H^*(t) := \sum_{\tau=t}^T -h'_i(b^*(\tau)) > 0$  and  $B^*(t) := \sum_{\tau=t}^T (\underline{b}^*(\tau) - \bar{b}^*(\tau))$ . Here,  $b^*(t)\underline{b}^*(t) = 0$  and  $(B - b^*(t))\bar{b}^*(t) = 0$  by complementary slackness.  $H^*(t)$  is the marginal storage cost-to-go at time  $t$ . If, e.g.,  $h(b) = \alpha(B - b)$  for some constant  $\alpha > 0$ , then  $H^*(t) = \alpha(T + 1 - t)$ .

Differentiating with respect to  $g(t)$ , the KKT condition implies  $(\tilde{b}(t) = \lambda(t))$

$$\gamma(t)g^*(t) = \tilde{b}^*(t) + \hat{\lambda}^*(t)$$

where  $\hat{\lambda}^*(t)g^*(t) = 0$ . Hence, from (13), the optimal solution is characterized by

$$g^*(t) = \left[ \bar{g}(t) + \frac{B^*(t)}{\gamma(t)} \right]^+ \quad (14)$$

$$b^*(t) = b^*(t-1) - d(t) + \left[ \bar{g}(t) + \frac{B^*(t)}{\gamma(t)} \right]^+ \quad (15)$$

where  $\bar{g}(t)$ , the nominal generation at time  $t$ , is defined as

$$\bar{g}(t) := \frac{H^*(t)}{\gamma(t)} \quad (16)$$

We now derive the structure of the optimal generation schedule  $g^*$  and the optimal battery energy level  $b^*$  from (14)–(16).

### B. Optimal solution

It is instructive to first look at the case where the battery constraint is inactive, i.e.,  $b(t) \in (0, B)$ . In this case,  $B^*(t) \equiv$

0, and the nominal generation schedule in (16) is optimal, i.e.,  $g^* = \bar{g}$ . This satisfies the condition

$$\gamma(t)g^*(t) = H^*(t) = \sum_{\tau=t}^T -\frac{dh}{db}(b^*(\tau))$$

The left-hand side is the marginal generation cost at time  $t$  and the right-hand side is the marginal storage cost-to-go from  $t$  to  $T$ . To balance the demand at time  $t$  under an optimal policy, one cannot reduce the cost further by increasing (decreasing) generation by a unit and raising (reducing) battery level by the same amount at time  $t$ . Moreover, if  $h'(b) < 0$ , the marginal storage cost-to-go, and therefore the marginal generation cost, strictly decreases in time under an optimal policy. These intuitions will generalize to the general case when the battery constraints may be active, as we now show.

Our first main result characterizes the optimal generation schedule  $g^*(t)$  under the assumptions

- A0:** For  $t = 1, \dots, T$ ,  $d(t) > 0$ ,  $\gamma(t) > 0$ . For  $b \geq 0$ ,  $h'(b) < 0$ .
- A1:** For  $t = 1, \dots, T$ ,  $\gamma(t)d(t) - \gamma(t+1)d(t+1) < -h'(b(t))$ .

Assumption A1 restricts how fast the product  $\gamma(t)d(t)$  can decrease. If the battery constraint is not active, then the optimal generation  $g^*$  will balance the marginal generation cost and the marginal storage cost-to-go. Hence, the optimal marginal generation cost  $\gamma(t)g^*$  will decrease at the same rate at which the marginal storage cost-to-go  $H^*(t)$  decreases, which is the marginal storage cost  $h'(b^*(t))$  at time  $t$ .

The case where the battery constraint is active is more complicated because the optimal generation must anticipate future starvation and saturation. As we will see below, if the product  $\gamma(t)d(t)$  decreases more slowly than marginal storage cost  $h'(b(t))$ , then the optimal generation  $g^*$  retains a simple global structure where  $g^*$  crosses demand  $d(t)$  at most once, from above, and the optimal battery level  $b^*$  is unimodal. Indeed, under assumption A1, the optimal generation schedule generally has three phases. In the first phase,  $g^*(t) > d(t)$  and the battery charges. In the second phase,  $g^*(t) = d(t)$  and the battery remains saturated at  $b^*(t) = B$ . In the third phase,  $g^*(t) < d(t)$  and the battery discharges. Degenerate cases where one or more of the phases are missing are possible, depending on problem parameters  $T$ ,  $b(0)$ ,  $\gamma(t)$ , and  $d(t)$ .

If the battery never saturates, the optimal schedule degenerates into (at most) two phases. If the battery does saturate, the optimal schedule has three phases, defined by the beginning time  $l+1$  and the end time  $m$  of the second phase, with  $0 < l \leq m \leq T$ , where  $b^*(t) = B$  for  $t = l, \dots, m$ . To describe the optimal generation  $g^*$  precisely, define the following time averages over the entire horizon and over the

initial and final phases

$$\bar{d} := \frac{1}{T} \sum_{t=1}^T d(t), \quad \bar{\gamma} := \left( \frac{1}{T} \sum_{t=1}^T \frac{1}{\gamma(t)} \right)^{-1}$$

$$\bar{d}_1 := \frac{1}{l} \sum_{t=1}^l d(t), \quad \bar{\gamma}_1 := \left( \frac{1}{l} \sum_{t=1}^l \frac{1}{\gamma(t)} \right)^{-1}$$

$$\bar{d}_3 := \frac{1}{T-m} \sum_{t=m+1}^T d(t), \quad \bar{\gamma}_3 := \left( \frac{1}{T-m} \sum_{t=m+1}^T \frac{1}{\gamma(t)} \right)^{-1}$$

Our main result implies that it is optimal to charge the battery initially and discharge it later.

*Theorem 1:* Suppose A0 and A1 hold and  $0 < b(0) < B$ . The optimal generation  $g^*$  crosses the demand curve  $d(t)$  at most once, from above, and therefore the optimal battery level  $b^*$  is unimodal. Moreover, they take one of the following two forms.

- 1) The battery never saturates, and the optimal generation schedule is

$$g^*(t) = \bar{g}(t) + \frac{\bar{\gamma}}{\gamma(t)} [\bar{d} - \sigma]^+$$

with

$$\sigma := \frac{1}{T} \sum_{t=1}^T \bar{g}(t) + \frac{b(0)}{T}$$

Moreover,  $b^*(T) = T[\sigma - \bar{d}]^+$ .

- 2) There exist times  $l$  and  $m$ , with  $0 < l \leq m \leq T$ , for which  $b^*(t) = B$  for  $t = l, \dots, m$ . Before time  $l+1$ , the battery charges and its energy strictly increases over time towards  $B$ . After time  $m$ , it discharges and its energy strictly decreases over time from  $B$ . The optimal generation  $g^*$  has three phases:

$$g^*(t) = \begin{cases} \bar{g}(t) - \frac{\bar{\gamma}_1}{\gamma(t)} (\sigma_1 - \bar{d}_1), & t = 1, \dots, l \\ d(t), & t = l+1, \dots, m \\ \bar{g}(t) + \frac{\bar{\gamma}_3}{\gamma(t)} [\bar{d}_3 - \sigma_3]^+, & t = m+1, \dots, T \end{cases}$$

where

$$\sigma_1 := \frac{1}{l} \sum_{t=1}^l \bar{g}(t) - \frac{B - b(0)}{l}$$

$$\sigma_3 := \frac{1}{T-m} \sum_{t=m+1}^T \bar{g}(t) + \frac{B}{T-m}$$

Moreover,  $b^*(T) = (T-m)[\sigma_3 - \bar{d}_3]^+$ .

Case 1 of Theorem 1 is illustrated in Figure 1. Here, the battery level remains strictly below  $B$  at all times. The threshold  $\sigma$  is the sum of average nominal generation and the initial battery energy per-unit-time. Under the nominal generation schedule, the total available energy over the entire period  $t = 1, \dots, T$  is exactly  $T\sigma$ . If the average demand  $\bar{d}$  is lower than the threshold  $\sigma$ , then the nominal generation is optimal,  $g^* = \bar{g}$ , and the terminal battery level can be positive. Otherwise, the nominal generation schedule will deplete the battery level to zero before time  $T$ , which is not optimal,

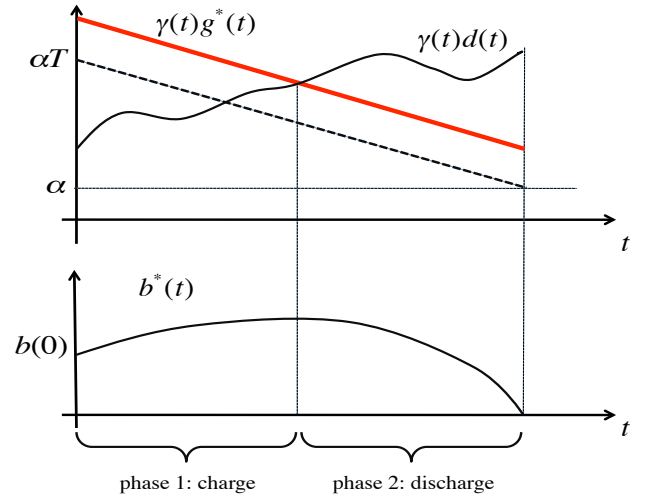


Fig. 1. Optimal marginal generation cost  $\gamma(t)g^*(t)$ , in comparison with  $\gamma(t)d(t)$  and battery level  $b^*(t)$ . This figure illustrates case 1 of Theorem 1 with  $\bar{d} > \sigma$ .

and hence the generation is shifted up by  $\bar{\gamma}(\bar{d} - \sigma)/\gamma(t)$  at each time  $t$ , so that the battery level hits zero exactly (and only) at time  $T$ .

Case 2 of Theorem 1 is illustrated in Figure 2. In this case,

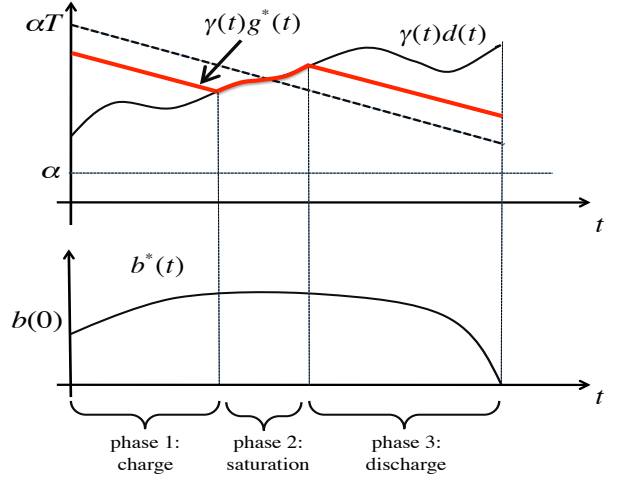


Fig. 2. Optimal marginal generation cost  $\gamma(t)g^*(t)$ , in comparison with  $\gamma(t)d(t)$ , and battery level  $b^*(t)$ . This figure illustrates case 2 of Theorem 1 with  $\bar{d}_1 < \sigma_1$  and  $\bar{d}_3 < \sigma_3$ .

the battery level rises from  $b(0)$  to  $B$  in phase 1, stays at  $B$  in phase 2, and drains in phase 3. The threshold  $\sigma_3$  is the sum of the average nominal generation over the third phase and the per-period battery level at the beginning of the third phase. If the nominal generation is used in the third phase, then the total available energy over this phase is exactly  $(T-m)\sigma_3$ . If the average demand  $\bar{d}_3$  over the third phase is less than the threshold  $\sigma_3$ , then the nominal generation is optimal in the third phase,  $g^* = \bar{g}$ , and the terminal battery level can be positive (this is similar to case 1 with initial battery level  $B$ , but over only the third phase). On the other hand, if  $\bar{d}_3 > \sigma_3$ , then the nominal generation in the third phase is shifted up

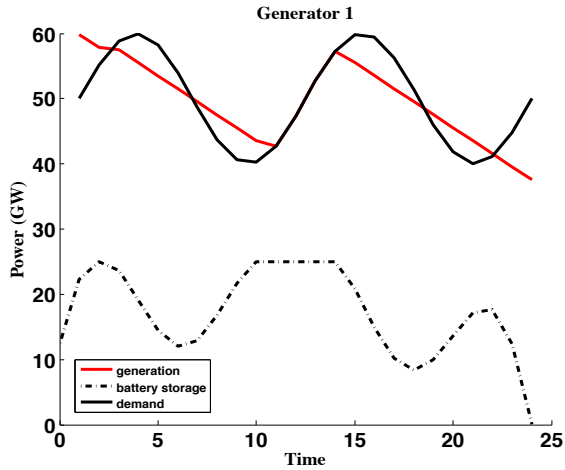


Fig. 3. Numerical results for a single generator single load when assumption A1 does not hold. Optimal generation  $g^*$  is linear when the battery charges and discharges, and matches demand when the battery is saturated.

in the third phase so that the battery level hits zero exactly at time  $T$ .

#### Example 1: SGSL case without assumption A1.

We present an SGSL example where assumption A1 does not hold. The horizon  $T = 24$  hours and each time step  $t$  represents 1 hour. The demand profile is

$$d(t) = 10 \cdot \sin \frac{4\pi}{T-1} (t-1)$$

Battery capacity  $B$  is 25 GW, with an initial battery level  $b(0)$  of 12.5 GW. We set  $\gamma(t) \equiv 1$  in the generation cost  $c$  and use  $h(b) = \alpha(B - b)$  with  $\alpha = 2$ . Figure 3 shows the numerical result for one horizon representing a 24-hour day. The optimal battery level  $b^*$  increases to saturation, but because demand decreases faster than that required for Assumption A1,  $b^*$  discharges and recharges twice more before reaching zero at the final time. The optimal generation  $g^*$  remains linear (because  $\gamma$  is time-invariant) when the battery charges and discharges, and follows demand when the battery is saturated, but does so several times over the entire control horizon.

#### C. Proof idea

In this subsection, we sketch the proof idea by describing some finer properties of the optimal solution. The detailed proofs are omitted due to space constraint.

The first lemma states that, under an optimal policy, the battery is never exhausted until possibly the last period. This is a consequence of the convexity of the cost function  $c(g, t)$  in  $g$ .

*Lemma 1:* Suppose A0 and A1 hold. Under an optimal policy,  $b^*(t) > 0$  for  $t = 1, \dots, T-1$ .

The next result describes the behavior in the third phase of Theorem 1. Once an optimal generation  $g^*(t)$  drops below the demand  $d(t)$ , it will remain strictly below the (time-varying) demand for subsequent periods. The battery therefore drains. Additionally, the optimal marginal cost

$\gamma(t)g^*(t)$  decreases at a rate equal to the marginal battery cost  $h'(b^*(t))$  at time  $t$ .

*Lemma 2:* Suppose A0 and A1 hold. If  $g^*(t_1) < d(t_1)$  for some  $t_1 \in \{1, \dots, T-1\}$  under an optimal policy, then

- 1)  $0 < g^*(t) < d(t)$ , for  $t = t_1, \dots, T$ . Moreover, for  $t = t_1, \dots, T-1$ ,

$$\gamma(t+1)g^*(t+1) = \left[ \gamma(t)g^*(t) + \frac{dh}{db}(b^*(t)) \right]^+.$$

- 2)  $b^*(t) < b^*(t-1)$  for  $t = t_1, \dots, T$ . Moreover, for  $t = t_1, \dots, T-1$ ,

$$b^*(t) - b^*(t+1) > \frac{\gamma(t)}{\gamma(t+1)} (b^*(t-1) - b^*(t)).$$

The next result describes the behavior in the first phase of Theorem 1. If an optimal generation exceeds the demand at a certain time, then it must have stayed strictly above the demand at all previous times. Therefore the battery must have been in the charging mode throughout this period.

*Lemma 3:* Suppose A0 and A1 hold. If  $b^*(t_0) < B$  and  $g^*(t_0+1) > d(t_0+1)$  for some  $t_0 \in \{1, \dots, T-1\}$  under an optimal policy, then

- 1)  $g^*(t) > d(t)$  for  $t = 1, \dots, t_0+1$ . Moreover, for  $t = 1, \dots, t_0$ ,

$$\gamma(t+1)g^*(t+1) = \gamma(t)g^*(t) + \frac{dh}{db}(b^*(t)).$$

- 2)  $b^*(t) > b^*(t-1)$  for all  $t = 1, \dots, t_0+1$ . Moreover, for  $t = 2, \dots, t_0+1$ ,

$$b^*(t) - b^*(t-1) < \frac{\gamma(t-1)}{\gamma(t)} (b^*(t-1) - b^*(t-2)).$$

Lemmas 2 and 3 imply that the optimal generation  $g^*$  can cross  $d(t)$  at most once, from above. Moreover, the optimal battery level is unimodal, i.e., it can only increase initially, until possibly reaching  $B$ . If the battery becomes saturated, it will remain saturated until the optimal generation drops below the demand, and can then only decrease for the remaining times. Theorem 1(1) describes the case where  $b^*$  is never saturated and Theorem 1(2) describes the case where  $b^*$  reaches saturation in phase two.

## IV. NETWORK CASE

We now consider a general network with multiple generators and multiple loads (i.e., the problem in (8)-(9)). To simplify, we eliminate the variables  $q_i(t)$  by combining (2), (3), and (4) into:

$$\begin{aligned} g_i(t) + r_i(t) &= \sum_{j \in \mathcal{N}} Y_{ij} (\theta_i(t) - \theta_j(t)), & i \in \mathcal{G} \\ -d_i(t) &= \sum_{j \in \mathcal{N}} Y_{ij} (\theta_i(t) - \theta_j(t)), & i \in \mathcal{D} \end{aligned}$$

Since the problem is convex, the KKT conditions are both necessary and sufficient. They imply the following characterization of the optimal generation

$$g^*(t) = \Gamma^{-1}(t) [H^*(t) + B^*(t)]^+ \quad (17)$$

where

$$H^*(t) = \text{diag} \left( \sum_{\tau=t}^T -\frac{dh_i}{db} (b_i^*(\tau)) \right)$$

$$B^*(t) = \text{diag} \left( \sum_{\tau=t}^T (b_i^*(\tau) - \bar{b}_i^*(\tau)) \right)$$

and  $\Gamma(t) = \text{diag}(\gamma_i(t))$ . Here,  $b_i^*(t)$  is the optimal battery level at generator  $i$ ,  $\underline{b}_i^*(t)$  and  $\bar{b}_i^*(t)$  are the Lagrange multipliers associated with  $b_i^*(t) = 0$  and  $b_i^*(t) = B_i$  at generator  $i$ . The optimal generation given in (17) is a directly extension of (14) and (16) from the SGSL case to the network case. The effect of network is illustrated most prominently in the following condition:

$$\begin{bmatrix} g^*(t) + r^*(t) \\ -d(t) \end{bmatrix} = \mathbf{Y}\theta^*(t) \quad (18)$$

where the network admittance matrix  $\mathbf{Y}$  is given by

$$\mathbf{Y}_{ij} = \begin{cases} \sum_{k \in \mathcal{N}} Y_{ik}, & i = j \\ -Y_{ij}, & i \neq j \end{cases}$$

We expect the macroscopic structure of the optimal policy to generalize from SGSL to the network case. For example, From (17), since the marginal storage cost-to-go  $H^*(t)$  decreases over time as long as  $h'_i(b_i) < 0$ , the optimal generation  $g^*(t)$  tends to decrease over time as in the SGSL case. Battery reserves are used to meet demands toward the end of the horizon, relying more on power generation at the the beginning. However, unlike in the SGSL case where the generation and the demand are directly related,  $g^*(t) + r^*(t) = d(t)$ , in the network case, they are indirectly related through the admittance matrix and the voltage angles as shown in (18). This complicates the proof and the properties of optimal policy in the network case are under current study. Here, we present some numerical examples.

### Example 2: Symmetric network with two generators.

The network consists of two generators and 20 demand nodes. Every generator is directly connected to each of the demand nodes, but there is no direct connection between the two generators nor among the demand nodes. The admittance on every link is  $Y_{ij} = 1$ . Link capacities  $q_{ij} = 1000$  are uniform and high.  $T = 24$ ,  $B_i = 16$  and  $b_i(0) = 8$ . The load profile is  $d_i(t) = 3 \sin(\pi t/T) + 1$ . The storage cost is  $h_i(b_i) = \alpha(B_i - b_i)$  with  $\alpha = 1$ ;  $\gamma_i(t) = 1$  in the generation cost function  $c_i$ .

The results are shown in Fig. 4. The optimal generation decreases when the battery initially charges and then tracks the net power output  $q_i(t)$  as the battery remains saturated. The optimal generation then becomes strictly less than the net power output and the battery discharges until the battery drains completely and the generation again tracks net power output.

### Example 3: Cost savings.

In this example, we consider both a SGSL case and a network case and illustrate how the time-varying nature of  $\gamma_i(t)$  in the

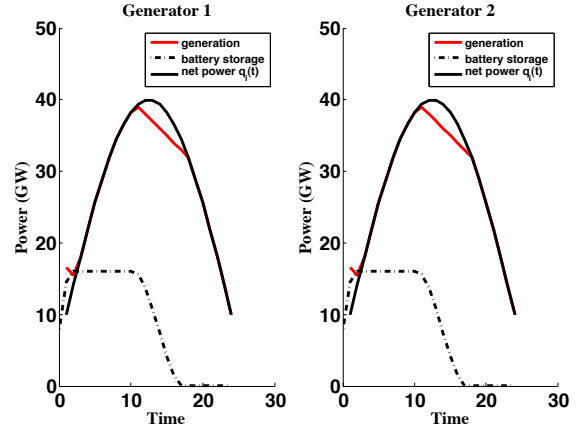


Fig. 4. Numerical results for the case with 2 generator nodes and 20 demand nodes. The configuration is symmetric, i.e. both generators are connected to every demand node, with equal admittance  $Y_{ij}$  for every  $i \in \mathcal{G}$  and  $j \in \mathcal{D}$ .

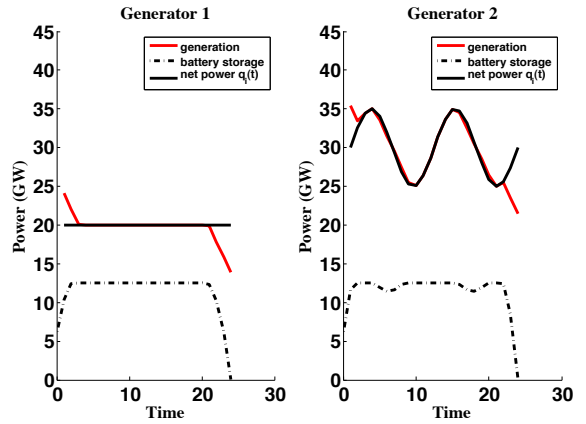


Fig. 5. Numerical results for the case with 2 generator nodes and 20 demand nodes with unequal link capacities. Demands vary sinusoidally.

generation cost can affect the cost saving. The network that is used is similar to that in Example 2 except that the link capacities are not symmetric: some of the link capacities between generator 1 and the demand nodes are smaller. The behavior is shown in Fig. 5. Interestingly, the optimal battery energy  $b^*(t)$  of generator 2 is not unimodal, i.e., it charges and discharges multiple times.

Without battery, the total generation must equal to total demand at each time. In particular, for SGSL,  $g^*(t) = d(t)$  for all  $t$  and the total cost is  $\sum_t c(d(t), t)$  without battery. With battery, the generation cost can be reduced through optimization over time. Table I compares the optimal cost in (8) as a percentage of total cost without battery, for SGSL/MGML and for time-invariant and time-varying  $\gamma_i$ . In the case with time-invariant  $\gamma_i$ , the cost savings is small. With time-varying  $\gamma_i$ , the cost savings in both the SGSL and the network cases are significantly higher. This confirms the intuition that battery is more valuable in the presence of fluctuations. The savings are greater in cases when the battery charges and discharges several times over the entire control period, which is a case that violates Assumption A1.

TABLE I  
OPTIMAL COSTS WITH BATTERIES AS % OF TOTAL COST WITHOUT  
BATTERIES

	SGSL	MGML
time-invariant $\gamma_i$	98%	97%
time-varying $\gamma_i$	83%	85%

Then with larger or more frequent fluctuations, an optimal policy has more opportunities to charge the batteries when generation costs are low and discharge when they are high.

## V. CONCLUSION

In this paper, we have formulated a simple optimal power flow model with storage. We have characterized the optimal solution for the case with a single generator and a single load when the generation cost  $c(g, t)$  is quadratic in  $g$  and the battery cost  $h(b)$  is strictly decreasing. In this case, under the assumption (A1) that the demand does not decrease too rapidly, the optimal generation schedule will cross the demand profile at most once, from above. The optimal battery level is unimodal where the battery is charged initially, possibly reaching saturation, and then discharges till the end of the control horizon. We have presented some numerical examples illustrating the behavior in the network case and when assumption A1 is not satisfied.

We plan to extend the analysis here in several ways. First, we will fully characterize the optimal solution in the network case by extending our results for the SGSL case. Second, both the demand profile and the power generation are deterministic in our current formulation, which is unrealistic. We will extend the current deterministic model to include randomness in demand and generation. Finally, we plan to incorporate a more realistic battery model that captures the dynamics of capacitance, as well as constraining the rate at which the battery can be charged or discharged.

## VI. ACKNOWLEDGMENTS

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