Consensus on Asynchronous Communication Networks in Presence of External Input

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***Abstract*—This paper presents a class of multi-agent systems where the state of each agent is driven by its own local protocol, and by exogenous time-varying input signal. These inputs may represent agent dynamics and, in presence of unreliable communication medium, transmission noise. We consider multi-agent systems operating over asynchronous networks. Examples of such systems are message-passing systems in which agents don't share a global clock and communicate only by sending and receiving messages that may be lost, delayed arbitrarily, and delivered out of order. We present conditions on the protocols and input signals that ensure zero or bounded steady-state error over asynchronous communication networks. We discuss two specific examples and apply our general results to them. These systems are an iterative group-based protocol for average consensus of time-varying quantities, and a scheme for spatial pattern configuration, where mobile agents communicate via message-passing.**

I. Introduction

We consider *dynamic* problems on multi-agent systems operating over unreliable networks. The goal of the agents is to track time-varying quantities spread throughout the system. These quantities may model time-varying sensor readings, such as local temperature or distance to a target, noise due to unreliable communication or agent dynamics, such as formations of autonomous mobile agents following threats. These problems arise in several contexts, such as load balancing [5], vehicle formation [7], [17], sensor fusion [16], [14], distributed coordination and flocking [10]. Examples of these problems are *dynamic average consensus* problems [19], where the goal of the agents is to track the average of time-varying inputs over switching topologies, and *dynamic pattern formation* problems [18], where agents aim to form and maintain some specific spatial configuration while leader agents move. In this problem, agents may operate over message-passing unreliable networks that allow for delayed, lost and out-of-order messages.

This class of dynamic consensus problems differ from the well-studied class of *static consensus* problems. In static consensus, the task of an agent is to estimate quantities that are constant throughout an execution [20]. Early work on such problems include [4], [6]. Recent theoretical results include work on discrete-time consensus [2], [10], asynchronous consensus [12], consensus over switching topologies [15], [13], [19], quantize consensus [11], and consensus over random networks [9], [22].

Unlike the static consensus problem, in dynamic consensus problems consensus is not constant; rather, it is determined by external inputs. Previous work on dynamic consensus includes [18], [21], [8], [23], [24]. In [18], authors provide conditions on input signals that ensure zero or bounded steady-state error for synchronous discrete and continuous-time linear schemes with fixed network topology. In [21], authors present an algorithm able to track the average of ramp inputs with zero steady-state error. The work in [8] investigates distributed iterative linear schemes for continuous-time consensus over switching topologies. They provide a distributed scheme that ensures bounded steady-state error when tracking bounded time-varying inputs with bounded derivatives. In [23], [24], authors focus on discrete-time algorithms for synchronous networks. They present a family of iterative schemes and prove they converge to the time-varying average for a general class of input functions, that includes polynomial, logarithmic-type functions and periodic functions.

In this paper, we extend the results of [18] to asynchronous systems. We present a general class of discrete-time protocols for asynchronous multi-agent systems and discuss their convergence properties. These protocols compute additive statistics of the system, such as average. We provide conditions on the protocols in absence of exogenous signals and conditions on the input signals that ensure zero or bounded steady-state error. Specifically, we can bound the steady-state error of the system when inputs are bounded, and, in absence of external inputs, the protocol is an additive function having zero steady-state error.

The organization of the paper is as follows. Section II describes a general discrete-time scheme for asynchronous unreliable networks in absence of exogenous inputs. Section III extends this by presenting discrete-time schemes for asynchronous networks with inputs. Therein, we provide conditions on the algorithms and input signals that ensure zero or bounded steady-state error. Section IV and Section V apply these results to two specific dynamic consensus protocols. Section IV presents bounds on the steady-state error of a group-based algorithm that solves the dynamic average consensus problem. In Section V, we derive bounds on the steady-state error of a general class of pattern formation schemes, where inputs to the system model dynamics of leader agents. Section VI concludes the paper.

II. PRELIMINARIES

We describe a general discrete-time protocol for asynchronous multi-agent systems in the absence of external inputs [1]. This section provides a foundation for later material, where external inputs are considered.

A system consists of N agents. We denote by I the set of agent indices $\{1, 2, \ldots, N\}$. The state-space of agent $i \in I$ is denoted X_i. The Cartesian product of agents is denoted X, where $X = \Pi_{i \in I} X_i$. Given $x \in X$, x_i is the i-th component of x. Associated with agent i is a set of actions, denoted by $\mathcal{F}_i$. Each individual action $f_i \in \mathcal{F}_i$ is a function, $f_i : X \to X_i$, that computes a new value for agent i using values of other agents.

The nature of asynchronous protocols, as considered herein, means that agents update their state with agent values that are old, or perhaps heterogenous with respect to when they were computed. To properly account for such anomalies, we build a time dynamic into our system model: for each pair of agents (i, j), let $\tau_{(i,j)} : \mathbb{N} \to \mathbb{N}$. Given a time value t, $\tau_{(i,j)}$ returns a time value less-than, or equal-to, t. With this in mind, the discrete-time protocol of agent i is

$$x_i(t) = \begin{cases} f_i(x_1(\tau_{(i,1)}(t-1)), \ldots, x_N(\tau_{(i,N)}(t-1))) & t > 0 \\ x0_i & t = 0 \end{cases} \tag{1}$$

where $x0$ is the initial vector. During execution, agent i chooses, and applies to its state, an action $f_i \in \mathcal{F}_i$. The input of f_i is a vector $(x_1(\tau_{(i,1)}(t-1)), \ldots, x_N(\tau_{(i,N)}(t-1)))$, where $x_j(\tau_{(i,j)}(t-1))$ is the value of x_j at time $\tau_{(i,j)}(t-1)$. This notation allows modeling message-passing systems operating over unreliable communication, where messages can be lost, delayed or received out-of-order. For a detailed description, see [1, Chapter 6].

A system action f is represented as a function $f : X \to X$ where

$$f(x) = (f_1(x), f_2(x), \ldots, f_N(x)) \tag{2}$$

with $f_i \in \mathcal{F}_i$. We denote by $\mathcal{F}$ the set of actions of the system.

Hence, the discrete-time asynchronous protocol of the system is

$$x(t) = \begin{cases} f(\ldots, f_i(x_1(\tau_{(i,1)}(t-1)), \ldots), \ldots) & t > 0 \\ x0 & t = 0 \end{cases} \tag{3}$$

A vector $x \in X$ is a *fixed point* of the iterative scheme in Eq. (3), if $x = f(x)$ for all $f \in \mathcal{F}$. We introduce the function $g : X \to X$, which maps each $x \in X$ to the fixed point of the iterative scheme in Eq. (3) having initial vector $x0 = x$. The function g has the form $g(x) = (g_1(x), g_2(x), \ldots, g_N(x))$ with $g_i : X \to X_i$.

Given the initial vector $x0$, we are interested in proving that $g(x0)$ is the solution to the problem and that it is an asymptotically stable equilibrium point. Throughout the paper, we denote by x^* the vector $g(x0)$. In static consensus problems we have that $\forall t,\ g(x(t)) = g(x0)$.

In order to prove asymptotically stability of x^*, we introduce the error function of the system $e : X \to \mathbb{R}^+$ as follows

$$e(x) = ||x - g(x)||_c \tag{4}$$

where $|| \cdot ||_c$ denotes the c-norm of the vector for some c. The vector x^* is an asymptotically stable equilibrium point, or equivalently the system converges to x^*, if the steady-state error of the system, denoted by $\hat{e}$, is 0, i.e.

$$\hat{e} = \lim_{t \to \infty} e(x(t)) = 0 \tag{5}$$

A sufficient condition for proving zero steady-state error can be derived from the following function. Given α, $0 \le \alpha < 1$, and C a positive constant, the step signal $\gamma_{\alpha,C} : \mathbb{N} \to \mathbb{R}^+$ is defined as

$$\gamma_{\alpha,C}(t) = \alpha^k C \quad t \in [t_k, t_{k+1}) \tag{6}$$

where $\{[t_k, t_{k+1}]\}_{k \ge 0}$ are consecutive time intervals called *epochs* with $t_0 = 0$. The following property ensures convergence of the iterative scheme to x^*.

Property 2.1: There exists a factor α, $0 \le \alpha < 1$ such that for all executions starting from $x0$, the sequences of errors of the execution are bounded above by the signal $\gamma_{\alpha,C}$ for some C depending on $x0$ and some $\{[t_k, t_{k+1}]\}_{k \ge 0}$ depending on the execution.

This property is a sufficient condition for proving that the system converges to x^*. Informally, the steady-state error of the system is 0, since the sequence $C, \alpha C, \alpha^2 C, \alpha^3 C, \ldots$ converges to 0 as time goes to infinity. Throughout this paper, we denote by l_k the length of epoch $[t_k, t_{k+1}]$.

III. ASYNCHRONOUS SYSTEM WITH EXTERNAL INPUTS

This section introduces external inputs, comparing the error of the system with and without them.

A. Dynamic Consensus

In the dynamic consensus case, each agent has a *reference* (or *input*) signal $r_i(t)$, with $r_i : \mathbb{N} \to X_i$. The protocol of agent i is updated to reflect this addition:

$$x_i(t) = \begin{cases} f_i(x_1(\tau_{(i,1)}(t-1)), \ldots) + r_i(t-1) & t > 0 \\ x0_i + r_i(0) & t = 0 \end{cases} \tag{7}$$

Note that if r_i is 0 for all t, Eq. (7) reduces to Eq. (1). We denote by $r : \mathbb{N} \to X$ the function defined as $r(t) = (r_1(t), r_2(t), \ldots r_N(t))$. In presence of external inputs, the error function has an analogous definition. When external inputs are added to the system, the consensus vector is *dynamic*, meaning that $\exists t : g(x(t)) \ne g(x0)$. In this section, we provide sufficient conditions that ensure that the error of the system e in presence of inputs is bounded.

B. Assumptions

The following assumptions on the action set allow us to bound the error on the system:

Assumption 3.1: Property 2.1 holds for protocol in Eq. (3).

Assumption 3.2: For all $i \in I$, $f_i \in \mathcal{F}_i$ is additive,

$$\forall x, r \in X : \ f_i(x + r) = f_i(x) + f_i(r) \tag{8}$$

Assumption 3.3: For all $i \in I$, g_i is additive,

$$\forall x, r \in X: \; g_i(x + r) = g_i(x) + g_i(r) \tag{9}$$

Assumptions 3.2 and 3.3 can be extended to all $f \in \mathcal{F}$ and g. Moreover, g's additivity has implications for the additivity of the error function:

Lemma 3.4: Function e is sub-additive,

$$\forall x, r \in X: \; e(x + r) \leq e(x) + e(r) \tag{10}$$

Proof: By definition (Eq. (4)), $e(x+r) = ||(x+r)-g(x+r)||_c$. Using Assumption 3.3, $e(x+r) = ||x-g(x)+r-g(r)||_c$. By the triangle inequality, $e(x + r) \leq ||x - g(x)||_c + ||r - g(r)||_c$. By definition, $e(x + r) \leq e(x) + e(r)$. ■

We can use these properties to separate the error of the system due to agent interactions from error due to exogenous input signals when describing system executions:

Lemma 3.5: $\forall x, r0, \ldots, rn \in X$, and $f^1, \ldots, f^n \in \mathcal{F}$,

$$e(\hat{x}) \leq e(f^n(\ldots(f^1(x)))) + \sum_{i=0}^{n} e(f^n(\ldots(f^{i+1}(ri)))) \tag{11}$$

where $\hat{x} = f^n(\ldots f^2(f^1(x + r0) + r1)\ldots) + rn$.

Proof: Follows from Assumptions 3.2 and 3.3 and Lemma 3.4. ■

As we will see in the next section, such manipulation is particularly important when bounding the error of the system.

C. Bound on the Execution Error

Consider two, generic, executions of the system: π, an execution in presence of external inputs; and $\bar{\pi}$ the same sequence of actions where all inputs are equal to 0. These two executions $\pi, \bar{\pi}$ apply the same sequence of actions to the same initial vector $x0$. We denote by πrt the fragment of $\bar{\pi}$ starting from time t; the execution πrt has initial vector $r(t)$ and no external inputs. We denote by e_π the error function e along the execution π, by $e_{\bar{\pi}}$ the error function e along the execution $\bar{\pi}$, and by $e_{\pi rt}$ the error function e along the execution πrt. in absence of external inputs

We have that an upper bound on the error of the system in presence of external inputs at the end of each epoch can be described as follows.

Theorem 3.6: If the system in absence of external inputs satisfies Assumptions 3.1, 3.2 and 3.3, then for all $[t_k, t_{k+1}]$ epochs of the system execution, $k \geq 0$,

$$e_\pi(x(t_{k+1})) \leq \alpha^{k+1} C + \sum_{j=0}^{k} \alpha^{k-j} l_j C^{(j)} \tag{12}$$

where $0 \leq \alpha < 1$, C satisfies Assumption 3.1 and $C^{(j)}$ is an upper bound on the error of the inputs injected in the epoch $[t_j, t_{j+1}]$ and l_j is the length of epoch $[t_j, t_{j+1}]$.

Proof: Given π, we can write $x(t_{k+1})$ as the iterative application of Eq. (7) starting from $x0$. Using Lemma 3.5, we have that the error of $x(t_{k+1})$ in the execution π is bounded above by the linear combination of the error of $x(t_{k+1})$ along the execution $\bar{\pi}$ and the sum of the errors of $x(t_{k+1} - t)$ along executions πrt,

$$e_\pi(x(t_{k+1})) \leq e_{\bar{\pi}}(x(t_{k+1})) + \sum_{t \leq t_{k+1}} e_{\pi rt}(x(t_{k+1} - t)) \tag{13}$$

Using Assumption 3.1, the error of $x(t_{k+1})$ in $\bar{\pi}$ is upper bounded as follows

$$e_{\bar{\pi}}(x(t_{k+1})) \leq \alpha^{k+1} C \tag{14}$$

The summation term of Eq. (13) can be partitioned into epochs:

$$\sum_{t \leq t_{k+1}} e_{\pi rt}(x(t_{k+1} - t)) \leq \sum_{j=0}^{k} \sum_{t \in [t_j, t_{j+1}]} e_{\pi rt}(x(t_{k+1} - t)) \tag{15}$$

$$\leq \sum_{j=0}^{k} l_j \max_{t \in [t_j, t_{j+1}]} e_{\pi rt}(x(t_{k+1} - t)) \tag{16}$$

where l_j is the length of the interval $[t_j, t_{j+1}]$. Using Assumption 3.1,

$$\max_{t \in [t_j, t_{j+1}]} e_{\pi rt}(x(t_{k+1} - t)) \leq \alpha^{k-j} C^{(j)} \tag{17}$$

for some $C^{(j)}$ such that $\gamma_{\alpha, C^{(j)}}$ is an upper bound on the error of πrt. Together,

$$e_\pi(x(t_{k+1})) \leq \alpha^{k+1} C + \sum_{j=0}^{k} \alpha^{k-j} l_j C^{(j)} \tag{18}$$

■

Given the previous upper bound, if the errors along all executions starting from input vectors are bounded, then the steady-state error is also bounded.

Corollary 3.7: If $\forall t_{k+1}, t$, $e_{\pi rt}(x(t_{k+1} - t))$ is uniformly bounded by $C^{(r)}$, then

$$\hat{e} \leq \frac{1}{1 - \alpha} \max_j l_j C^{(r)} \tag{19}$$

Proof: It follows from Theorem 3.6 by taking the limit of the formula in Eq. 12. ■

Convergence to the equilibrium point is reached asymptotically in the following special case.

Corollary 3.8: If $\forall t$, $r(t)$ is a fixed point of the system, then $\hat{e} = 0$.

Proof: It follows because, by definition $f(r(t)) = r(t)$ for all $f \in \mathcal{F}$ and $e(r(t)) = 0$. ■

D. Discussion

Our results apply to problems having additive solutions (see Assumption 3.3) and hold for additive protocols with zero steady-state error in absence of exogenous inputs (see Assumption 3.2 and 3.1). In particular, Assumption 3.1 requires linear converge to the solution in absence of inputs with rate α. This assumption is satisfied if there exists a sequence of stable monotonic sets around x^* such that the system moves monotonically between sets in finite time while its error decreases by α. Examples of algorithms that satisfy our assumptions are linear schemes that estimate linear system statistics, such as linear schemes for average consensus problems.

The bound on the error at the end of epoch k, presented in Theorem 3.6, consists of the sum of two terms. The first term is a bound on the error of the system at the end of epoch k in absence of external inputs. The second term is a linear combination of bounds, each of them is an upper bound on the error at the end of epoch k of exogenous inputs added at epoch j, for all $j \leq k$. These two terms depend on α because, by assumption, the system converges linearly with rate α.

As shown in Corollary 3.7, the steady-state error of the system in presence of external inputs is bounded if inputs are uniformly bounded by $C^{(r)}$. This bound depends on $\alpha, C^{(r)}$ and the length of the epochs. If the error of all inputs is 0, then the steady-state error is 0 (see Corollary 3.8). Notice that a system can be driven by non-zero external inputs while having a steady-state error of zero.

Our results apply to discrete time protocols, but similar results can also be derived in the case of continuous time schemes.

IV. Average Consensus

The first example presents an iterative group-based protocol for solving the dynamic average consensus problem. This protocol has been studied before both with [18] and without [10], [3] inputs.

In Subsection IV-A we describe the static average consensus problem and a scheme for solving it. In Subsection IV-B, we show that Assumptions 3.1, 3.2, and 3.3 hold. In Subsection IV-C we describe the dynamic average consensus problem and derive bounds on the error of the system applying Theorem 3.6.

A. Static Consensus Problem

The system consists of N agents, each storing a real number, $X_i = \mathbb{R}$. The goal of the system is to compute the average of the initial values in a distributed fashion.

We consider group-based local protocols where each agent of the group set its value to a linear combination of its current value and the average of the group. This linear combination is weighted by a factor denoted by β. The agents that do not belong to the set are unchanged. Denoting by A the group, i the agent, the action $f_{A,i,\beta}$ is defined as

$$f_{A,i,\beta}(x) = \begin{cases} (1-\beta)x_i + \beta \frac{1}{|A|}\sum_{j \in A} x_j & i \in A \\ x_i & i \notin A \end{cases} \tag{20}$$

The protocol of agent i becomes

$$x_i(t) = \begin{cases} f_{A,i,\beta}(x(t-1)) & t > 0 \\ x0_i & t = 0 \end{cases} \tag{21}$$

where $x \in \mathbb{R}^N$. As shown in the previous equation, in this example we consider a system where agents update their state using the current state of other agents.

The set of actions of agent i is $\mathcal{F} = \{f_{A,i,\beta}\}$ where $i \in I$, $A \subseteq I$ and $L \leq \beta \leq 1$, with L being some positive constant.

Given A and β, the action set of the system $\mathcal{F} = \{f_{A,\beta}\}$, with

$$f_{A,\beta}(x) = (f_{A,1,\beta}(x), f_{A,2,\beta}(x), \ldots, f_{A,N,\beta}(x)) \tag{22}$$

The goal of the agents is to compute the average of the initial vector $x0$ using this decentralized local scheme. Fixed points of this algorithm are vectors having all equal values. The solution vector to the static average consensus problem, i.e. the vector storing the average of $x0$ in each entry, is a fixed point of the system.

The function g is defined as $g(x) = (g_1(x), \ldots, g_N(x))$, with $\forall i, j$, $\forall x \in X$, $g_i(x) = g_j(x)$. The function g_i with $i \in I$ is defined as follows, $\forall x \in X$

$$g_i(x) = \frac{1}{|N|}\sum_{j \in I} x_j \tag{23}$$

B. Properties of the System

As proved in [10], [3], the protocol converges to the average of the initial values for all system executions having no permanent group partitions. In [3], we show that the error function e, defined as 2-norm error, satisfies Property 2.1 with α defined as

$$\alpha = \sqrt{1 - \frac{2L(1-L)}{N(N-1)^2}} \tag{24}$$

and $C = e(x0)$. We briefly discuss the structure of epochs for this problem. Given a sequence of actions a, we can construct an undirected graph $G(a) = (V, E)$ with $V = I$ and $\{i, j\} \in E$ if there exists an action $f_{A,k,\beta} \in a$ such as $i, j \in A$. Given an execution π, the $\{[t_k, t_{k+1}]\}_{k \geq 0}$ of π satisfies the following property: $\forall k$, $G(\pi_{[t_k, t_{k+1}]})$ is connected with $\pi_{[t_k, t_{k+1}]}$ being the sub-sequence of actions of π from time t_k to t_{k+1} [18].

We next prove that functions $f_{A,i,\beta}$, for all i, A, β, and g_i, for all $i \in I$, are additive.

Lemma 4.1: For all $i \in I$, $A \subseteq I$, $0 \leq \beta \leq 1$, $f_{A,i,\beta}$ is additive. Formally,

$$\forall x, r \in \mathbb{R}^N \quad f_{A,i,\beta}(x + r) = f_{A,i,\beta}(x) + f_{A,i,\beta}(r) \tag{25}$$

Proof: We distinguish two cases depending on whether or not i is in A:
If $i \in A$, $f_{A,i,\beta}(x + r)$

$$= (1-\beta)(x_i + r_i) + \beta \frac{1}{|A|}\sum_{j \in A}(x_j + r_j) \tag{26}$$

$$= \left((1-\beta)x_i + \beta\frac{1}{|A|}\sum_{j \in A} x_j\right) + \left((1-\beta)r_i + \beta\frac{1}{|A|}\sum_{j \in A} r_j\right) \tag{27}$$

$$= f_{A,i,\beta}(x) + f_{A,i,\beta}(r) \tag{28}$$

by definition of $f_{A,i,\beta}(x)$ and $f_{A,i,\beta}(r)$.

If $i \notin A$, then $f_{A,i,\beta}(x + r) = x + r$. Noticing that $x = f_{A,i,\beta}(x)$ and $r = f_{A,i,\beta}(r)$, the lemma follows. ∎

In the next lemma we show that g_i is additive.

Lemma 4.2: For all $i \in I$, g_i is additive

$$\forall x, r \in \mathbb{R}^N \; g_i(x + r) = g_i(x) + g_i(r) \tag{29}$$

Proof: By definition,

$$g_i(x+r) = \frac{1}{|N|}\sum_{j\in I}(x_j + r_j) \tag{30}$$

$$= \frac{1}{|N|}\sum_{j\in I} x_j + \frac{1}{|N|}\sum_{j\in I} r_j. \tag{31}$$

Noticing that, $g_i(x) = \frac{1}{|N|}\sum_{j\in I} x_j$ and $g_i(r) = \frac{1}{|N|}\sum_{j\in I} r_j$, the lemma follows. ■

C. Dynamic Consensus

In presence of external inputs, values are injected into the systems. The protocol of each agent becomes,

$$x_i(t) = \begin{cases} f_{A,i,\beta}(x(t-1)) + r_i(t-1) & t > 0 \\ x0_i + r_i(0) & t = 0 \end{cases} \tag{32}$$

The error of the system at the end of the k-th epoch of an execution π in presence of inputs can be bounded as follows.

Theorem 4.3: For all $[t_k, t_{k+1}]$ epochs of the system execution,

$$e_\pi(t_{k+1}) \leq \alpha^{k+1} e(x0) + \sum_{j=0}^{k} \alpha^{k-j} l_j \max_{t\in[t_j,t_{j+1}]} e(r(t)) \tag{33}$$

Proof: It follows from Theorem 3.6, because Assumptions 3.1, 3.2 and 3.3 are satisfied. Assumption 3.1 has been proved in [3], with $C = e(x0)$ and α defined in Eq. (24). Assumptions 3.2 and 3.3 follow from Lemmas 4.1 and 4.2. Hence,

$$e_\pi(x(t_{k+1})) \leq \alpha^{k+1} C + \sum_{j=0}^{k} l_j \alpha^{k-j} C^{(j)} \tag{34}$$

where $C = e(x0)$. Noting that for all x, f, $e(f(x)) \leq e(x)$, we have that $C^{(j)} = \max_{j\in[t_j,t_{j+1}]} e(r(t))$. ■

Hence, we can bound the steady-state error of the system as follows.

Corollary 4.4: If $r(t)$ is uniformly bounded, then the steady state error is upper bounded

$$\hat{e} \leq \frac{1}{1-\alpha} \max_j l_j \max_t e(r(t)) \tag{35}$$

Proof: Follows from Theorem 4.3 and Corollary 3.7. ■

In the special case when the input vector is time-invariant we have that the previous bound becomes

Corollary 4.5: If $\forall t$, $r(t) = b$ for some constant vector b, then the steady-state error is bounded by

$$\hat{e} \leq \frac{1}{1-\alpha} \max_j l_j e(b) \tag{36}$$

Proof: Follows from Corollary 3.7. ■

Finally, there are cases when in presence of external inputs the steady-state error is 0. As shown in [18], if agents are driven by a common input, then the steady-state error is 0.

Corollary 4.6: If $\forall t$, $\forall i, j :\ r_i(t) = r_j(t)$, then $\hat{e} = 0$.

Proof: It follows from Corollary 3.8, since, by assumption, $\forall t$, $r(t)$ is a fixed point of the system. ■

V. Linear Pattern Formations

In this section, we provide bounds on the steady-state error of a class of pattern formation protocols in presence of external inputs. These protocols can be modeled as iterative schemes for solving systems of linear equations. In Subsection V-A, we present the static problem and iterative schemes for solving it. In Subsection V-B, we prove that the Assumptions 3.1, 3.2, and 3.3 hold. In Subsection V-C we we add exogenous inputs to the schemes and derive bounds on their steady-state errors applying Theorem 3.6. Finally, in Subsection V-D we apply the results to pattern formation protocols.

A. Static Systems of Linear Equations

The system consists of N agents. The goal of the system is to solve the system of linear equations, $Ax = b0$, with A being an invertible real-valued matrix of size $N \times N$ and $b0$ a real-valued vector of length N. Agents solve the problem by executing the Gauss algorithm starting from an initial guess $x0$ in a decentralized way where agent i is responsible for equation i of the system.

The state of agent i consists of the pair (x_i, b_i) with $X_i = \mathbb{R}^2$. The action of agent i can be represented as a function $f_i : X \rightarrow X_i$ as follows:

$$f_i((x,b)) = \left(b_i - \sum_{j\neq i} A(i,j)x_j, b_i\right) \tag{37}$$

where, without loss of generality, we assume that $A(i,i) = 1$ for all $i \in I$.

The protocol of agent i is

$$(x_i, b_i)(t) = \begin{cases} (f_i((x_1(\tau_{(i,1)}(t-1)), \ldots), b)) & t > 0 \\ (x0_i, b0_i) & t = 0 \end{cases} \tag{38}$$

This protocol models a message-passing system operating over unreliable communication [1, Chapter 6] where agents send and receive messages.

The fixed points of the system are pairs $(x, b) \in X$ such that $x_i = (A^{-1}b)_i$ for all $i \in I$. The $(A^{-1}b)_i$ denotes the i-th component of vector $A^{-1}b$. The function g_i is defined as

$$g_i((x,b)) = ((A^{-1}b)_i, b_i) \tag{39}$$

and the function $g : X \rightarrow X$ is defined as $g((x,b)) = (g_1((x,b)), \ldots, g_N((x,b)))$. Notice that the solution of $Ax = b0$ is a fixed point of the system.

In this problem, we are interested in the error of the first component of the state of the system,

$$e((x,b)) = ||(x,b)_1 - g(x,b)_1||_c \tag{40}$$

B. Properties of the System

In earlier work [3], we provide conditions for ensuring that steady-state error of the system is 0. We prove that the ∞-norm error of the system satisfies Property 2.1 assuming that

A is weakly diagonally dominant, i.e.

$$\forall i \in I \quad : \quad \sum_{j \neq i} |A(i,j)| \leq A(i,i) \tag{41}$$

$$\exists k \in I \quad : \quad \sum_{j \neq k} |A(k,j)| < A(k,k) \tag{42}$$

and the communication delay along any communication link is bounded by some unknown constant b, i.e.

$$\forall i, j \in I, t \in \mathbb{N} \;\; t - b \leq \tau_{(i,j)}(t) \leq t \tag{43}$$

In [3], the α factor is computed as follows. Given the incidence graph $G = (V, E)$ corresponding to A, defined as $V = I$ and $(i, j) \in E$ if $A(i, j) \neq 0$, we construct a forest of trees $\mathbb{F}$ rooted at nodes satisfying Eq. (42). The decreasing factor α is defined using $\mathbb{F}$.

Lemma 5.1 ([3]): If A is weakly diagonally dominant, then ∞-norm error e satisfies Property 2.1 with α being $\alpha = ||w||_\infty$ where

$$w(j) = \begin{cases} \sum_{k \neq j} |A(j,k)| & j \textit{ root of } \mathbb{F} \\ |A(j,p(j))| \; w(p(j)) + \sum_{k \neq \{j,p(j)\}} |A(j,k)| \end{cases} \tag{44}$$

where $p(j)$ is the parent of j in $\mathbb{F}$, $C = e(x0)$ and $\{[t_k, t_{k+1}]\}_{k \geq 0}$ depends on the execution.

We briefly discuss the epochs of this system. Given an execution π, we denote by $\pi_{[t_k,t_{k+1}]}$ the sequence of actions of π from time t_k to time t_{k+1}. The sequence of $\{[t_k, t_{k+1}]\}_{k \geq 0}$ of π is defined as follows. Given $k \geq 0$, there exists an increasing sequence of times $t^1, \ldots, t^N \in [t_k, t_{k+1}]$, such as for all $i \in I$, $t^i > t^{p(i)}$ and agent i updates its state at time t^i using a value for the parent computed at a time greater or equal-to $t^{p(i)}$. Using the bounded delay assumption on the communication channels, we have that the epochs of all executions of the system are uniformly bounded by $B = b \cdot h(\mathbb{F})$, with $h(\mathbb{F})$ being the height of $\mathbb{F}$.

We next show that f_i and g_i are additive.

Lemma 5.2: For all $i \in I$, f_i is additive

$$\begin{aligned} &\forall (x, b), (xr, xb) \in X \\ &f_i((x, b) + (xr, br)) = f_i((x, b)) + f_i((xr, br)) \end{aligned} \tag{45}$$

Proof: By definition

$$f_i((x{+}xr, b{+}br)) = (b_i{+}br_i{-}\sum_{j \neq i} A(i,j)(x_j{+}xr_j), b_i{+}br_i) \tag{46}$$

Distributing, $f_i((x + xr, b + br))$ is equal to

$$(b_i - \sum_{j \neq i} A(i,j)x_j, b_i) + (br_i - \sum_{j \neq i} A(i,j)xr_j, br_i) \tag{47}$$

The lemma follows noticing that

$$f_i((x, b)) \;\; = \;\; (b_i - \sum_{j \neq i} A(i,j)x_j, b_i) \tag{48}$$

$$f_i((xr, br)) \;\; = \;\; (br_i - \sum_{j \neq i} A(i,j)xr_j, br_i) \tag{49}$$

■

We next consider g_i,

Lemma 5.3: For all $i \in I$, the function g_i is additive

$$\begin{aligned} &\forall (x, b), (xr, xb) \in X \\ &g_i((x, b) + (xr, br)) = g_i((x, b)) + g_i((xr, br)) \end{aligned} \tag{50}$$

Proof: It follows by algebraic manipulation,

$$\begin{aligned} g_i((x + xr, b + br)) &= ((A^{-1}(b + br))_i, (b + br)_i) \\ &= ((A^{-1}b)_i, b_i) + ((A^{-1}br)_i, br_i) \\ &= g_i((x, b)) + g_i((xr, br)) \end{aligned} \tag{51}$$

■

C. Systems of Linear Equation with Inputs

The protocol of agent i when there are external inputs is defined as follows

$$(x_i, b_i)(t) = \begin{cases} f_i((x_1(\tau_{(i,1)}(t-1)), \ldots), b) + r_i(t-1) & t > 0 \\ (x0_i, b0_i) + r_i(0) & t = 0 \end{cases} \tag{52}$$

where $r_i : \mathbb{N} \to X_i$ is the input function of agent i. In presence of inputs, both vector b and x can be modified.

Using Theorem 3.6, we can bound the error of the system at the end of each epoch.

Theorem 5.4: For all $[t_k, t_{k+1}]$ epochs of the system execution,

$$e_\pi(x(t_{k+1})) \leq \alpha^{k+1} C + B \sum_{j=0}^{k} \alpha^{k-j} C^{(j)} \tag{53}$$

where $C = e(x0)$ and $C^{(j)} = \max_{t \in [t_j, t_{j+1}]} e(r(t))$.

Proof: Follows from Lemmas 5.1, 5.2 and 5.3 and Theorem 3.6. ■

In the case of steady-state error, we have that

Corollary 5.5: If $e(r(t))$ is uniformly bounded, then

$$\hat{e} \leq \frac{1}{1 - \alpha} B \max_t e(r(t)) \tag{54}$$

Proof: It follows from Corollary 3.7. ■

If inputs have a special structure, then the steady-state error of the system is 0.

Corollary 5.6: If $\forall t \in \mathbb{N}, \exists b \in \mathbb{R}^N$ such that $r(t) = (A^{-1}b, b)$, then $\hat{e} = 0$.

Proof: It follows from Corollary 3.8, since, by assumption, $\forall t$, $r(t)$ is a fixed point of the system. ■

Hence, we have convergence in presence of inputs, if for all agents, the first component of the inputs is a linear combination of the second components weighted using matrix A.

D. Pattern formation Protocols

Linear formation problems can be cast into systems of linear equation problems of the form $Ax = b$, where vector x represents agent positions, and vector b together with A defines the target formation of the multi-agent system and corresponding algorithm for reaching it.

In our model, the state of the system is driven by the algorithm and external inputs. These inputs can change the value in the x and b vectors. An exogenous signal added to x

models changes in agent positions, that can be due to agent dynamics or communication noise. An input added to b models a change in the algorithm and in the target formation.

In the special case of pattern formation problems we are interested in two input signals, $\forall t$, $r1(t) = (\hat{x}, \mathbf{0})$ and $r2(t) = (\mathbf{0}, \hat{b})$, where $\hat{x}, \hat{b} \in \mathbb{R}^N$ and $\mathbf{0}$ is a vector having all entries equal to 0. Signal $r1$ models a system where the target formation is time-invariant, while messages containing agent positions may be forged. Signal $r2$ models a system where the formation changes with time, but no communication error is allowed.

In the case when the state of the system is driven by the algorithm and input signal $r1$, we get the following bound on the steady-state error of the system.

Corollary 5.7: If $\forall t$, $r(t) = r1(t)$ then

$$\hat{e} \leq \frac{1}{1-\alpha} B ||\hat{x}||_\infty \tag{55}$$

Proof: It follows from Corollary 5.5. ■

When the state is driven by $r1$, we have that steady-state error is depends on α, B and on the maximum allowed transmitting noise ($||\hat{x}||_\infty$). Instead, in the case when the state is driven by $r2$, we have that values of the target formation move with constant velocity, given by the vector $\hat{b}$, and we derive the following bound.

Corollary 5.8: If $\forall t$, $r(t) = r2(t)$ then

$$\hat{e} \leq \frac{1}{1-\alpha} B ||A^{-1}\hat{b}||_\infty \tag{56}$$

Proof: It follows from Corollary 5.5. ■

In this case, the steady-state error depends on α, B and agent velocities.

VI. Conclusions

We have provided convergence analysis for a general class of dynamic consensus algorithms operating over asynchronous switching networks. This class includes additive protocols used for computing additive statistics of the system. We have shown that this class of protocols has bounded steady-state errors if the exogenous signals are bounded and the protocol, in the absence of inputs, has zero steady-state error. We have also presented conditions that ensure zero steady-state error in presence of external inputs. We have discussed the applicability of our results to a dynamic average consensus protocol and dynamic pattern formation protocol. In the case of average consensus, we have shown that the system converges to the time-varying average if agents have a common input. In the case of pattern formation, we have proved that the system has bounded steady-state error if the formation moves with constant velocity.

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